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Uniqueness of Minimal Projections in Smooth Matrix Spaces

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Let $X = (M(n, m), \|\cdot\|)$, where $\|\cdot\|$ fulfills Condition 0.3 and $W = M(n, 1) + M(1, m)$. A formula for a minimal projection from X onto W is given in (E. W. Cheney and W. A. Light, 1985, “Approximation Theory in Tensor Product Spaces,” Lecture Notes in Mathematics, Springer-Verlag, Berlin; E. J. Halton and W. A. Light, 1985, *Math. Proc. Cambridge Philos. Soc.* **97**, 127–136; and W. A. Light, 1986, *Math. Z.* **191**, 633–643). We will show that this projection is the unique minimal projection (see Theorem 2.1).

Key Words: approximation theory; minimal projection; uniqueness of minimal projection; tensor product spaces; Orlicz spaces; smooth spaces.

0. INTRODUCTION

Let $M(n, m)$ be the space of all (real or complex) matrices of n rows and m columns. Denote by $M(n, 1)$ ($M(1, m)$, respectively) the space of matrices from $M(n, m)$ with constant rows (constant columns respectively). Let S_n be the group of permutations of the set $\{1, 2, ..., n\}$.

Definition 0.1. Put $S_n \times S_m = \{\pi = \sigma \times \eta : \sigma \in S_n \text{ and } \eta \in S_m\}$. $S_n \times S_m$ is a group with a natural operation. Consider the transformations $A_{\sigma \times \eta}$ on $M(n, m)$ associated with $S_n \times S_m$, i.e,

$$(A_{\sigma \times \eta} x)(i, j) = x(\sigma(i), \eta(j)). \tag{0.1}$$

Definition 0.2. Let X be a Banach space X. An element $x \in X$ is called a *smooth point* if it has a unique supporting functional f_x. If every x from a unit sphere is a smooth point, then X is called a *smooth* space.

For some basics facts of smoothness as well as some interesting applications see, e.g., [14].

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Throughout this paper, unless otherwise stated, we will assume that the norm $\|\cdot\|$ on $M(n, m)$ fulfills the following condition.

Condition 0.3. (1) For any $\pi \in S_n \times S_m$ the transformation A_π (see Definition 0.1) is an isometry.

(2) Space $(M(n, m), \|\cdot\|)$ is smooth.

Now we present some general facts concerning minimal projections.

Let $\mathscr{P}(X, W)$ denote the set of all continuous linear projections from X onto W, i.e.,

$$\mathscr{P}(X, W) = \{P \in \mathscr{L}(X, W) : P/_{\mathscr{W}} = Id_W\}.$$

A projection $P_0 \in \mathscr{P}(X, W)$ is called *minimal* if

$$\|P_0\| = \lambda(W, X) = \inf\{\|P\| : P \in \mathscr{P}(X, W)\}.$$

The constant $\lambda(W, X)$ is called the *relative projection constant*.

The method for proving the uniqueness of a minimal projection in our case is based on the two well-known theorems which earlier have only been used for proving minimality of projections. We take advantages of both these theorems combined with smoothness of the considered space. The author hopes that this method could be useful in other cases.

The first of these theorems comes from W. Rudin (Theorem 0.6) and the second from B. L. Chalmers and F. T. Metcalf (Theorem 0.8). Before we state these theorems we have to introduce some notions and definition.

DEFINITION 0.4. Suppose that a Banach space X and a topological group G are related in the following manner: to every $s \in G$ corresponds a continuous linear operator T_s: $X \to X$ such that

$$T_e = I, \qquad T_{st} = T_s T_t \quad (s \in G, t \in G).$$

Under these conditions, G is said to act as a group of linear operators on X.

DEFINITION 0.5. A map L: $X \to X$ commutes with G if $T_g L T_{g^{-1}} = L$ for every $g \in G$.

THEOREM 0.6 [23, III.B.13]. *Let X be a Banach space and W a complemented subspace, i.e., $\mathscr{P}(X, W) \neq \emptyset$. Let G be a compact group which acts as a group of linear operators on X such that*

(1) *$T_g(x)$ is a continuous functions of g, for every $x \in X$,*

(2) *$T_g(W) \subset W$, for all $g \in G$.*

Then for every $\varepsilon > 0$ there exists a projection P: $X \to W$ which commutes with G such that $\|P\| \leqslant (\lambda(W, X) + \varepsilon) \sup_{g \in G} \|T_g\|^2$.

Fix any projection Q from X onto W such that $\|Q\| \leqslant \lambda(W, X)+\varepsilon$ then the desired projection P is defined by

$$P(x)=\int_G T_g Q T_{g^{-1}}(x)\, dg, \qquad \textit{for} \quad x \in X,$$

where dg denotes the normalized Haar measure on G.

In many concrete applications T_g are isometries and the projection which commutes with G is unique. Then this theorem implies that the norm of this projection equals $\lambda(W, X)$; thus this projection is minimal. It does not imply that this projection is the unique minimal projection as there could be projections which do not commute with G but still have a minimal norm. For applications of the above theorem and related results see, e.g., [2, 11–13, 15–19, 21, 23].

Below we assume that X is a normed space and W is a finite-dimensional subspace.

DEFINITION 0.7. A pair $(x, y) \in S(X^{**}) \times S(X^*)$ will be called an *extremal pair* for $P \in \mathscr{P}(X, W)$ if $y(P^{**}x)=\|P\|$, where $P^{**}: X^{**} \rightarrow W$ is the second adjoint extension of P to X^{**} (S denotes unit sphere). Let $\mathscr{E}(P)$ be the set of all extremal pairs for P.

To each $(x, y) \in \mathscr{E}(P)$ associate the rank-one operator $y \otimes x$ from X to X^{**} given by $(y \otimes x)(z)=y(z) \cdot x$ for $z \in X$.

THEOREM 0.8 [8, Theorem 1]. *A projection $P \in \mathscr{P}(X, W)$ has a minimal norm if and only if the closed convex hull of $\{y \otimes x\}_{(x, y) \in \mathscr{E}(P)}$ contains an operator E_P for which W is an invariant subspace.*

The operator E_P is given by the formula

$$E_P=\int_{\mathscr{E}(P)} y \otimes x\, d\mu(x, y): X \rightarrow X^{**},$$

where μ is a probabilistic Borel measure on $\mathscr{E}(P)$.

For applications of Theorem 0.8 see, e.g., [3–9, 22].

Now, we recall one simple fact concerning the trace of a linear operator.

DEFINITION 0.9 [23, III.F.26]. If $L: X \rightarrow X$ (X finite dimensional) is a linear operator then L has a representation (non-unique of course) in the form $L(x)=\sum_i x_i^*(x)\, x_i$. The trace of L is defined as

$$tr(L)=\sum_i x_i^*(x_i).$$

It is well known that this definition is correct, i.e., it does not depend on a particular representation of L.

A nice sketch of applications of the trace as well as some further results could be found in [23, Section III.F].

1. PRELIMINARY RESULTS

Consider the group $G=S_n\times S_m$, to every $\pi=\sigma\times\eta\in S_n\times S_m$ coincides the transformation A_π (see Definition 0.1). Then $G=S_n\times S_m$ acts as a group of linear operators on $M(n,m)$ (see Definition 0.4).

Let $X=(M(n,m),\|\cdot\|)$, where $\|\cdot\|$ fulfills Condition 0.3 and $W=(n,1)+M(1,m)$.

The following two results were presented in [12, Chapter 9].

THEOREM 1.1 [12]. *For any $\pi\in S_n\times S_m$ the subspace W is invariant under A_π, i.e., $A_\pi(W)\subset W$.*

THEOREM 1.2 [12]. *There is the unique projection $Q\colon X\to W$ which commutes with $G=S_n\times S_m$ given by the formula*

$$Qe_{rs}(i,j)=\begin{cases}\dfrac{n+m-1}{nm} & i=r,\quad j=s\\[2ex] \dfrac{m-1}{nm} & i\neq r,\quad j=s\\[2ex] \dfrac{n-1}{nm} & i=r,\quad j\neq s\\[2ex] \dfrac{-1}{nm} & i\neq r,\quad j\neq s,\end{cases} \tag{1.1}$$

where $e_{rs}(i,j)=\delta_{ri}\delta_{sj}$.

The group $S_n\times S_m$ meets the assumptions of Theorem 0.6. Thus the projection given by (1.1) is minimal.

DEFINITION 1.3. Let $P\colon X\to W$ be a projection. Put

$$\mathscr{E}_0(P)=\{(x,y)\in S(X)\times S(X^*): y(Px)=\|P\|\}. \tag{1.2}$$

To each $(x,y)\in\mathscr{E}_0(P)$ associate the rank-one operator $y\otimes x$ from X to X given by $(y\otimes x)(z)=y(z)\cdot x$ for $z\in X$.

Since X and W are finite dimensional $\mathscr{E}_0(P)$ is non-empty. Now, we prove a very useful lemma.

LEMMA 1.4. *Take any $y \in X^*$ and $\pi \in S_n \times S_m$. Then $y(A_\pi^{-1}(z)) = (A_\pi y)(z)$, for every $z \in X$. (X is finite dimensional, so algebraically X^* can be treated as X).*

Proof. Since X is finite dimensional, each $y \in X^*$ can be represented as

$$y(x) = \sum_{i,j} y_{i,j} \cdot x_{i,j},$$

where $x = \sum_{i,j} x_{i,j} \cdot e_{i,j}$ and $y_{i,j} \in \mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) are independent of x.

Therefore, since $A_{\sigma\times\eta}^{-1} = A_{\sigma^{-1}\times\eta^{-1}}$,

$$\begin{aligned} y(A_\pi^{-1}(z)) &= \sum_{i,j} y_{i,j} \cdot (A_{\sigma\times\eta}^{-1}(z))_{i,j} \\ &= \sum_{i,j} y_{i,j} \cdot (A_{\sigma^{-1}\times\eta^{-1}}(z))_{i,j} \\ &= \sum_{i,j} y_{i,j} \cdot z_{\sigma^{-1}(i),\eta^{-1}(j)} \\ &= \sum_{i,j} y_{\sigma(i),\eta(j)} \cdot z_{i,j} \\ &= \sum_{i,j} (A_{\sigma\times\eta}(y))_{i,j} \cdot z_{i,j} \\ &= (A_\pi y)(z). \quad \blacksquare \end{aligned}$$

THEOREM 1.5. *If $(x, y) \in \mathscr{E}_0(Q)$ then $(A_\pi x, A_\pi y) \in \mathscr{E}_0(Q)$, for every $\pi \in S_n \times S_m$.*

Proof. By Theorem 1.2, Q commutes with $S_n \times S_m$, i.e., $Q = (A_\pi)^{-1} Q A_\pi$ for every $\pi \in S_n \times S_m$. Hence, by Lemma 1.4

$$\begin{aligned} \|Q\| = y(Qx) &= y((A_\pi)^{-1} Q A_\pi(x)) \\ &= y((A_\pi)^{-1} (Q A_\pi(x))) \\ &= (A_\pi y)(Q(A_\pi x)). \quad \blacksquare \end{aligned}$$

We now define an operator crucial to our further considerations. Take any $(x, y) \in \mathscr{E}_0(Q)$ and put

$$E_Q = \frac{1}{|G|} \sum_{\pi \in G} (A_\pi y) \otimes (A_\pi x) \colon X \to X. \tag{1.3}$$

(Compare this to Definition 1.3 and Theorem 0.8.) Since $(y \otimes x)^* = x \otimes y$ then

$$E_Q^* = \frac{1}{|G|} \sum_{\pi \in G} (A_\pi x) \otimes (A_\pi y) \colon X \to X. \tag{1.4}$$

Take the following elements of W:

$$\begin{aligned} &u_r \in M(n, m), \text{ for } r = 1, ..., n: && u_r(i, j) = \begin{cases} 0 & i \neq r \\ 1 & i = r \end{cases}; \\ &v_s \in M(n, m), \text{ for } s = 1, ..., m: && v_s(i, j) = \begin{cases} 0 & j \neq s \\ 1 & j = s \end{cases}; \\ &w \in M(n, m): && w(i, j) = 1. \end{aligned} \tag{1.5}$$

Since $\dim W = n + m - 1$, the elements $u_1, ..., u_{n-1}, v_1, ..., v_{m-1}, w$ form a basis of W.

THEOREM 1.6. *Using the above definitions*:

(1) $E_Q(W) \subset W$.

(2) $E_Q^*(w) = c \cdot w$.

Proof. Take any u_r $(r = 1, ..., n)$. We will show that $E_Q(u_r) \in W$. By Lemma 1.4

$$\begin{aligned} |G| \cdot E_Q(u_r) &= \sum_\pi (A_\pi y) \otimes (A_\pi x)(u_r) \\ &= \sum_\pi (A_\pi y)(u_r) \cdot A_\pi(x) \\ &= \sum_\pi y(A_\pi^{-1}(u_r)) \cdot A_\pi(x) \\ &= \sum_\pi y(A_{\pi^{-1}}(u_r)) \cdot A_\pi(x) \\ &= \sum_{\pi'} y(A_{\pi'}(u_r)) \cdot A_{(\pi')^{-1}}(x) \\ &= \sum_{\pi'} y(A_{\pi'}(u_r)) \cdot A_{\pi'}^{-1}(x). \end{aligned}$$

In the second to last equality we changed the summation, i.e, we put $\pi^{-1} = \pi'$.

Let $\pi(r, t) = \{\pi = \sigma \times \eta : \sigma(t)\}$. Now, we will continue our computation.

$$\begin{aligned}
|G| \cdot E_Q(u_r) &= \sum_\pi y(A_\pi(u_r)) \cdot A_\pi^{-1}(x) \\
&= \sum_{t=1, \ldots, n} \left(\sum_{\pi \in \pi(r, t)} y(A_\pi(u_r)) \cdot A_\pi^{-1}(x) \right) \\
&= \sum_{t=1, \ldots, n} \left(\sum_{\pi \in \pi(r, t)} y(u_t) \cdot A_\pi^{-1}(x) \right) \\
&= \sum_{t=1, \ldots, n} y(u_t) \cdot \left(\sum_{\pi \in \pi(r, t)} A_{\pi^{-1}}(x) \right) \\
&= \sum_{t=1, \ldots, n} y(u_t) \cdot \left(\sum_{\pi' \in \pi(t, r)} A_{\pi'}(x) \right). \qquad (1.6)
\end{aligned}$$

In the last equality we changed the summation, i.e., we put $\pi^{-1} = \pi'$ and we used the fact that $\pi \in \pi(r, t)$ if and only if $\pi^{-1} \in \pi(t, r)$. Consider the term in brackets in the equality above. Note that

$$\begin{aligned}
\left(\sum_{\pi \in \pi(t, r)} A_\pi(x) \right)(k, l) &= \sum_{\sigma \times \eta \in \pi(t, r)} x(\sigma(k), \eta(l)) \\
&= \sum_{\sigma : \sigma(t) = r} \left(\sum_\eta x(\sigma(k), \eta(l)) \right) \\
&= \sum_{\sigma : \sigma(t) = r} (m-1)! \left(\sum_{p=1, \ldots, m} x(\sigma(k), p) \right).
\end{aligned}$$

One can see that the term in the last equality does not depend on l. Hence

$$\left(\sum_{\pi \in \pi(t, r)} A_\pi(x) \right) \in W, \qquad \text{for any} \quad t = 1, \ldots, n.$$

Combining this with (1.6) gives $E_Q(u_r) \in W$ (for any $r = 1, \ldots, n$). Reasoning in a way similar to that above, but applied to v_s, leads to $E_Q(v_s) \in W$ (for any $s = l, \ldots, m$). Therefore the proof of (1) is complete.

By (1.4) and Lemma 1.4 we get

$$\begin{aligned}
|G| \cdot E_Q^*(w) &= \sum_\pi (A_\pi x) \otimes (A_\pi y)(w) \\
&= \sum_\pi (A_\pi x)(w) \cdot A_\pi(y)
\end{aligned}$$

$$= \sum_{\pi} x(A_{\pi}^{-1}(w)) \cdot A_{\pi}(y)$$

$$= \sum_{\pi} x(w) \cdot A_{\pi}(y)$$

$$= x(w) \cdot \left(\sum_{\pi} A_{\pi} \right) (y).$$

Since

$$\left(\sum_{\pi} A_{\pi} \right) (y) = (n-1)!\,(m-1)! \left(\sum_{i,j} y(i,j) \right) \cdot w,$$

the point (2) follows from the previous computation. ∎

THEOREM 1.7. *E_Q commutes with $S_n \times S_m$, i.e, $A_{\pi} E_Q = E_Q A_{\pi}$, for every $\pi \in S_n \times S_m$.*

Proof. Fix $\kappa \in S_n \times S_m$. By Lemma 1.4

$$\begin{aligned}
|G| \cdot E_Q \circ A_{\kappa}(v) &= \sum_{\pi} (A_{\pi} y) \otimes (A_{\pi} x)(A_{\kappa} v) \\
&= \sum_{\pi} (A_{\pi} y)(A_{\kappa} v) \cdot A_{\pi}(x) \\
&= \sum_{\pi} (A_{\kappa}^{-1} A_{\pi} y)(v) \cdot A_{\pi}(x) \\
&= \sum_{\pi} (A_{\kappa^{-1} \circ \pi} y)(v) \cdot A_{\pi}(x) \\
&= \sum_{\pi'} (A_{\pi'} y)(v) \cdot A_{\kappa \circ \pi'}(x) \\
&= \sum_{\pi'} (A_{\pi'} y)(v) \cdot A_{\kappa}(A'_{\pi})(x) \\
&= A_{\kappa} \left(\sum_{\pi'} (A_{\pi'} y)(v) \cdot (A'_{\pi})(x) \right) \\
&= A_{\kappa}(|G| \cdot E_Q(v)) \\
&= |G| \cdot A_{\kappa} \circ E_Q(v).
\end{aligned}$$

In the fifth equality in the above reasoning we changed the summation, i.e., we put $\kappa^{-1} \circ \pi = \pi'$. ∎

THEOREM 1.8. *If an operator L: $W \to W$ commutes with $S_n \times S_m$ ($A_\pi L = LA_\pi$) then there are constants a, b, k such that*

$$L(u_r) = au_r + \frac{k-a}{n} w, \qquad \textit{for} \quad i = 1, ..., n-1$$

$$L(v_s) = bu_s + \frac{k-b}{m} w, \qquad \textit{for} \quad i = 1, ..., m-1 \tag{1.7}$$

$$L(w) = kw.$$

Moreover, for any constants a, b, k the operator L defined by (1.7) commutes with $S_n \times S_m$.

Proof. Take $u_1, ..., u_{n-1}, v_1, ..., v_{m-1}, w$, which form a basis of W (see (1.5)). Any operator L: $W \to W$ can be represented in a general form as below

$$L(u_r) = \left(\sum_{k=1, ..., n-1} a_{kr} u_k \right) + \left(\sum_{l=1, ..., m-1} b_{lr} v_l \right) + e_r w,$$

$$\text{for any} \quad r = 1, ..., n-1$$

$$L(v_s) = \left(\sum_{k=1, ..., n-1} c_{ks} u_k \right) + \left(\sum_{l=1, ..., m-1} d_{ls} v_l \right) + f_s w, \tag{1.8}$$

$$\text{for any} \quad s = 1, ..., m-1$$

$$L(w) = \left(\sum_{k=1, ..., n-1} g_k u_k \right) + \left(\sum_{l=1, ..., m-1} h_l v_l \right) + tw.$$

The proof is divided into a few steps. Now, fix p, $q \in \{1, ..., n-1\}$ and i, $j \in \{1, ..., m-1\}$. Consider the transformation A such that

$$A(u_p) = u_q, \quad A(u_q) = u_p, \quad A(u_r) = u_r \qquad (\text{for } r \neq p, q),$$

$$A(v_i) = v_j, \quad A(v_j) = v_i, \quad A(v_s) = v_s \qquad (\text{for } s \neq i, j), \tag{1.9}$$

$$A(w) = w,$$

i.e., $A = A_{\sigma \times \eta}$ where $\sigma(k) = k$, for $k \neq p$, q and $\sigma(p) = q$, $\sigma(q) = p$, $\eta(l) = l$, for $l \neq i$, j and $\eta(i) = j$, $\eta(j) = i$. Since L commutes with so chosen A then $A \circ L(u_p) = L \circ A(u_p)$ which after substitution for L from (1.8) and for A from (1.9) yields

$$\left(\sum_{k \neq p, q} a_{kp} u_k \right) + a_{pp} u_q + a_{qq} u_p + \left(\sum_{l \neq i, j} b_{lp} v_l \right) + b_{ip} v_j + b_{jp} v_i + e_p w$$

$$= \left(\sum_{k \neq p, q} a_{kq} u_k \right) + a_{pq} u_p + a_{qq} u_q + \left(\sum_{l \neq i, j} b_{lp} v_l \right) + b_{iq} v_i + b_{jq} v_j + e_q w.$$

Comparing coefficients at the suitable elements of basis $u_1, ..., u_{n-1}, v_1, ..., v_{m-1}, w$ we get

$$\begin{aligned} &a_{pp}=a_{qq}, \quad a_{pq}=a_{qp}, \quad a_{kp}=a_{kq} \qquad \text{for any} \quad k\neq p, q, \\ &b_{ip}=b_{jq}, \\ &e_p=e_q. \end{aligned} \tag{1.10}$$

By the similar reasoning applied to the equality $A\circ L(v_i)=L\circ A(v_i)$

$$\begin{aligned} &c_{pi}=c_{qj}, \\ &d_{ii}=d_{jj}, \quad d_{ij}=d_{ji}, \quad d_{li}=d_{lj} \qquad \text{for any} \quad l\neq i, j. \end{aligned} \tag{1.11}$$

In addition the equality $A\circ L(w)=L\circ A(w)$ is equivalent to

$$\begin{aligned} &\left(\sum_{k\neq p,q} g_k u_k\right)+g_p u_q+g_q u_p+\left(\sum_{l\neq i,j} h_l v_l\right)+h_i v_j+h_j v_i+tw \\ &\quad =\left(\sum_{k\neq p,q} g_k u_k\right)+g_q u_p+g_q u_q+\left(\sum_{l\neq i,j} h_l v_l\right)+h_i v_i+h_j v_j+tw. \end{aligned}$$

Hence

$$g_p=g_q \qquad \text{and} \qquad h_i=h_j. \tag{1.12}$$

Therefore L has a form

$$\begin{aligned} &L(u_r)=a'\cdot\left(\sum_{k\neq r} u_k\right)+au_r+b\cdot\left(\sum_{l=1,...,m-1} v_l\right)+ew, \\ &\qquad \text{for any} \quad r=1, ..., n-1 \\ &L(v_s)=c\cdot\left(\sum_{k=1,...,n-1} u_k\right)+d'\cdot\left(\sum_{l\neq s} v_l\right)+dv_s+fw \\ &\qquad \text{for any} \quad s=1, ..., m-1 \\ &L(w)=g\cdot\left(\sum_{k=1,...,n-1} u_k\right)+h\cdot\left(\sum_{l=1,...,m-1} v_l\right)+tw. \end{aligned} \tag{1.13}$$

Now, consider the following transformation A

$$\begin{aligned} &A(u_i)=\left(w-\sum_{k=1,...,n-1} u_k\right), \quad A(u_r)=u_r \qquad (\text{for } r\neq i), \\ &A(v_i)=\left(w-\sum_{l=1,...,m-1} v_l\right), \quad A(v_s)=v_s \qquad (\text{for } s\neq j), \\ &A(w)=w, \end{aligned} \tag{1.14}$$

i.e., $A = A_{\sigma\times\eta}$ where $\sigma(k)=k$, for $k\neq i$, n and $\sigma(i)=n$, $\sigma(n)=i$, $\eta(l)=l$, for $l\neq j$, m and $\eta(j)=m$, $\eta(m)=j$. Since L commutes with so chosen A then $A\circ L(u_r)=L\circ A(u_r)$ (for $r\neq i$) which after substitution for L from (1.13) and for A from (1.14) yields

$$a'\cdot\left(\sum_{k\neq r} u_k\right)+au_r+a'\cdot\left(w-\sum_{k=1}^{n-1} u_k\right)+b\cdot\left(\sum_{l\neq j} v_l\right)+b\cdot\left(w-\sum_{l=1}^{m-1} v_l\right)ew$$

$$=a'\cdot\left(\sum_{k\neq r} u_k\right)+au_r+b\cdot\left(\sum_{l=1}^{m=1} v_l\right)+ew$$

which, after simplification, gives

$$(a-a')\,u_r-a'u_i+bv_j+(a'+b+e)\,w$$

$$=a'\cdot\left(\sum_{k\neq r} u_k\right)+au_r+b\cdot\left(\sum_{l=1}^{m-1} v_l\right)+ew.$$

Hence,

$$a'=0 \qquad \text{and} \qquad b=0. \tag{1.15}$$

Applying the same reasoning to the equality $A\circ L(v_s)=L\circ A(v_s)$ (for $s\neq j$) we also obtain

$$d'=0 \qquad \text{and} \qquad c=0. \tag{1.16}$$

In addition, from the equality $A\circ L(w)=L\circ A(w)$ we get

$$g\cdot\left(\sum_{k\neq i} u_k\right)+g\cdot\left(w-\sum_{k=1}^{n-1} u_k\right)+h\cdot\left(\sum_{l\neq j} v_l\right)+h\cdot\left(w-\sum_{l=1}^{m-1} v_l\right)tw$$

$$=g\cdot\left(\sum_{k=1}^{n-1} u_k\right)+h\cdot\left(\sum_{l=1}^{m-1} v_l\right)+tw,$$

and after simplification of the left side of the above inequality

$$gu_i+hv_j+tw=g\cdot\left(\sum_{k=1}^{n-1} u_k\right)+h\cdot\left(\sum_{l=1}^{m-1} v_l\right)+tw.$$

Hence

$$g=0 \qquad \text{and} \qquad h=0. \tag{1.17}$$

Comparing (1.15), (1.16), and (1.17) with (1.13) yields

$$\begin{aligned} &L(u_r)=au_r+ew, &&\text{for any} \quad r=1, ..., n-1\\ &L(v_s)=dv_s+fw, &&\text{for any} \quad s=1, ..., m-1\\ &L(w)=tw. \end{aligned} \tag{1.18}$$

Now, for A described in (1.14), from the equality $A\circ L(u_i)=L\circ A(u_i)$ we get

$$a\cdot\left(w-\sum_{k=1}^{n-1} u_k\right)+ew=tw-\sum_{k=1}^{n-1}(au_k+ew).$$

By comparing the coefficients at the element w in both sides of the above equality, $a+e=t-(n-1)\,e$, that is

$$e=\frac{t-a}{n}. \tag{1.19}$$

Applying the equality $A\circ L(v_j)=L\circ A(v_j)$, after similar computations we finish at

$$f=\frac{t-d}{m}. \tag{1.20}$$

Observe that (1.19) and (1.20) with (1.18) finished the proof of the first part of this theorem. It is left to show that every L given by (1.7) commutes with $S_n\times S_m$. To do this fix any $A_{\sigma\times\eta}$, where $\sigma\times\eta\in S_n\times S_m$. Simple computation using the form of L (see (1.7)) yields

$$\begin{aligned} L(u_n)&=L\left(w-\sum_{k=1}^{n-1} u_k\right)=tw-L\left(\sum_{k=1}^{n-1} u_k\right)\\ &=tw-\sum_{k=1}^{n-1} L(u_k)\\ &=tw-\sum_{k=1}^{n-1}\left(au_k+\frac{t-a}{n}w\right)\\ &=tw-a\cdot\left(\sum_{k=1}^{n-1} u_k\right)-(n-1)\,\frac{t-a}{n}\,w\\ &=tw-a(w-u_n)-(n-1)\,\frac{t-a}{n}\,w\\ &=au_n-\frac{t-a}{n}\,w. \end{aligned} \tag{1.21}$$

Therefore, for any $r = 1, ..., n$

$$A_\pi \circ L(u_r) = A_\pi \left(au_r - \frac{t-a}{n} w \right) = au_{\sigma(r)} - \frac{t-a}{n} w = L(A_\pi(u_r)) = L \circ A_\pi(u_r).$$

Since we can obtain the same equality as in (1.21) but for the element v_m and do the same computation as above for v_s, where $s = 1, ..., m$, the proof is completed. ∎

THEOREM 1.9. *There is a constant c such that $E_Q/_W = c \cdot Id_W$. Moreover $c = \|Q\|$.*

Proof. From Theorem 1.6 point (1) and Theorem 1.7 the operator $E_Q/_W$ fulfills the assumptions of Theorem 1.8. Therefore there are constants a, b, k such that E_Q can be represented in the form (1.7). Hence

$$(E_Q/_W)^* (w) = \left(\sum_{r=1, ..., n-1} \frac{k-a}{n} u_r \right) + kw + \left(\sum_{s=1, ..., m-1} \frac{k-b}{m} u_s \right),$$

and by Theorem 1.6 point (2) $\frac{k-a}{n} = 0$ and $\frac{k-b}{m} = 0$. Therefore $k = a = b$ which gives the first part of this theorem.

To prove the second part we will use the notion of a trace of an operator (for the necessary definition see Definition 1.3). Since

$$\begin{aligned} E_Q/_W \circ Q(v) &= \frac{1}{|G|} \sum_{\pi \in G} (A_\pi y) \otimes (A_\pi x)(Qv) \\ &= \frac{1}{|G|} \sum_{\pi} (A_\pi y)(Qv) \cdot A_\pi(x) \\ &= \frac{1}{|G|} \sum_{\pi} (A_\pi y \circ Q)(v) \cdot A_\pi(x), \end{aligned}$$

then by the definition of a trace of the operator $E_Q/_W \circ Q$, and Theorem 1.5 we get

$$tr(E_Q/_W \circ Q) = \frac{1}{|G|} \sum_{\pi} (A_\pi y \circ Q)(A_\pi(x)) = \|Q\|. \tag{1.22}$$

Let $z_1, ..., z_{nm}$ be a basis of X such that $z_1, ..., z_{n+m-1}$ form a basis of W. We can represent E_Q as follows

$$E_Q(x) = \sum_{k=1, ..., nm} z_k^*(x)\, z_k.$$

By Theorem 1.6 $E_Q(W) \subset W$, hence $z_k^*/_W = 0$, or $k = n+m+2, ..., mn$ and therefore

$$E_Q/_W(x) = \sum_{k=1, ..., n+m+1} z_k^*(x)\, z_k. \tag{1.23}$$

Since the trace of an operator does not depend on a particular representation we get

$$\begin{aligned} tr(E_Q/_W \circ Q) &= \sum_{k=1, ..., n+m+1} z_k^*(Qz_k) \\ &= \sum_{k=1, ..., n+m+1} z_k^*(z_k) \\ &= tr(E_Q/_W). \end{aligned} \tag{1.24}$$

From the first part of this theorem $E_Q/_W = c \cdot Id_W$, and since $tr(c \cdot Id_W) = c$ by (1.22) and (1.24) $c = \|Q\|$. ∎

2. MAIN RESULTS

The aim of this paper is to prove the following result.

THEOREM 2.1. *Let $X = (M(n, m), \|\cdot\|)$, where $\|\cdot\|$ fulfills Condition* 0.3 *and $W = M(n, 1) + M(1, m)$. Take the projection Q given by the formula* (1.1). *Then Q is the unique minimal projection from X to W.*

Proof. The projection Q is minimal. Assume, on the contrary, that there is a projection $P \neq Q$ with $\|P\| = \|Q\|$. Consider the projection

$$R = \left(\frac{1}{|G|} \sum_{\pi \in G} (A_\pi^{-1}) \circ P \circ A_\pi \right): X \to W.$$

Since, for any $\kappa \in S_n \times S_m$

$$\begin{aligned} A_\kappa \circ \left(\frac{1}{|G|} \sum_\pi (A_\pi^{-1}) \circ P \circ A_\pi \right) &= \frac{1}{|G|} \sum_\pi (A_\kappa \circ A_\pi^{-1}) \circ P \circ A_\pi \\ &= \frac{1}{|G|} \sum_\pi (A_{\kappa \circ \pi^{-1}}) \circ P \circ A_\pi \\ &= \frac{1}{|G|} \sum_{\pi'} A_{\pi'} \circ P \circ A_{\pi'^{-1} \circ \kappa} \end{aligned}$$

$$= \frac{1}{|G|} \sum_{\pi''} (A_{\pi''^{-1}}) \circ P \circ A_{\pi'' \circ \kappa}$$

$$= \frac{1}{|G|} \sum_{\pi''} (A_{\pi''}^{-1}) \circ P \circ (A_{\pi''} \circ A_{\kappa})$$

$$= \left(\frac{1}{|G|} \sum_{\pi} (A_{\pi}^{-1}) \circ P \circ A_{\pi} \right) \circ A_{\kappa},$$

the projection R commutes with $S_n \times S_m$, thus by virtue of Theorem 1.2 $R = Q$, i.e.,

$$Q = \frac{1}{|G|} \sum_{\pi \in G} (A_{\pi}^{-1}) \circ P \circ A_{\pi}. \tag{2.1}$$

For any projection R: $X \to W$ put (compare this with Definition 1.3)

$$\tilde{\mathscr{E}}_0(R) = \{(x, y) \in S(X) \times S(X^*) : |y(Rx)| = \|P\|\}. \tag{2.2}$$

Now, we will prove two lemmas

LEMMA 1. *If* $(u, v) \in \tilde{\mathscr{E}}_0(Q)$ *then* $(u, v) \in \tilde{\mathscr{E}}_0(P)$. ($P$, Q *are projections defined at the beginning of the proof.*)

Assume to the contrary that $|v(Pu)| < \|P\|$. Then by (2.1)

$$\|Q\| = |v(Qu)| = \left| v\left(\frac{1}{|G|} \sum_{\pi} (A_{\pi}^{-1}) \circ P \circ A_{\pi}(u) \right) \right|$$

$$= \left| \frac{1}{|G|} \sum_{\pi} v((A_{\pi}^{-1}) \circ P \circ A_{\pi}(u)) \right|$$

$$\leqslant \frac{1}{|G|} \sum_{\pi} |v((A_{\pi}^{-1}) \circ P \circ A_{\pi}(u))|. \tag{2.3}$$

Since $\|v\|_{X^*} = 1$, $\|u\|_X = 1$ and A_{π} are isometries,

$$|v((A_{\pi}^{-1}) \circ P \circ A_{\pi}(u))| \leqslant \|P\|, \qquad \text{for any} \quad \pi \in S_n \times S_m.$$

For $\pi = Id \times Id$

$$|v((A_{\pi}^{-1}) \circ P \circ A_{\pi}(u))| = |v(Pu)| < \|P\|.$$

Combining the two above estimations with (2.3) gives $\|Q\| < \|P\|$ and the desired contradiction.

LEMMA 2. *If* $(u, v) \in \mathscr{E}_0(Q)$ *then* $v \circ Q = v \circ P$. *(*P, Q *are projections defined at the beginning of the proof.)*

By Lemma 1

$$v(Qu) = \|Q\| \qquad \text{and} \qquad |v(Pu)| = \|P\|. \tag{2.4}$$

Hence, by $\|P\| = \|Q\|$, there is a $\alpha \in \mathbb{K}$: $|\alpha| = 1$ such that

$$\frac{v \circ Q}{\|Q\|}(u) = 1 \qquad \text{and} \qquad \alpha \cdot \frac{v \circ P}{\|Q\|}(u) = 1. \tag{2.5}$$

Since, by (2.4), $\|v \circ Q\| = \|v \circ P\| = \|Q\|$ and smoothness of $(X, \|\cdot\|)$ (see Definition 0.2 and Condition 0.3), in virtue of (2.5) we get

$$\frac{v \circ Q}{\|Q\|} = \alpha \cdot \frac{v \circ P}{\|Q\|}. \tag{2.6}$$

The above equality considered on the element $w = Q(u)$ yields $1 = \alpha$, which by (2.6) gives $v \circ Q = v \circ P$.

Now, by having Lemma 2 it is easy to finish the proof of Theorem 2.1.

Take the operator E_Q given by (1.3). Since $P \neq Q$ there is $x_0 \in X$ such that

$$w_0 := P(x_0) - Q(x_0) \neq 0. \tag{2.7}$$

Applying Lemma 2 to $(A_\pi x, A_\pi y) \in \mathscr{E}_0(Q)$ (see Theorem 1.5) yields

$$\begin{aligned}(A_\pi y)(w_0) &= (A_\pi y)(P(x_0) - Q(x_0)) \\ &= (A_\pi y)(P(x_0)) - (A_\pi y)(Q(x_0)) \\ &= ((A_\pi y) \circ P - (A_\pi y) \circ Q)(x_0) = 0.\end{aligned}$$

From the definition of E_Q and the above computation we get

$$\begin{aligned}E_Q(w_0) &= \frac{1}{|G|} \sum_{\pi \in G} (A_\pi y) \otimes (A_\pi x)(w_0) \\ &= \frac{1}{|G|} \sum_{\pi} (A_\pi y)(w_0) \cdot A_\pi(x) = 0.\end{aligned}$$

Since $w_0 \in W$, we have $E_Q(w_0) = \|Q\| \cdot w_0$ by Theorem 1.9; thus by (2.8) we have $w_0 = 0$; which contradicts (2.7). ∎

Below we present examples of norms $\|\cdot\|$ on $M(n, m)$ fulfilling Condition 0.3.

Since, in [12] it is proved that A_π are isometries in $(M(n, m), \|\cdot\|_p)$ for $p \in (1, \infty)$ and it is well known that $(M(n, m), \|\cdot\|_p)$ are smooth for $p \in (1, \infty)$ then we have the following

THEOREM 2.2. *If* $X = (M(n, m), \|\cdot\|_p)$ *for* $p \in (1, \infty)$ *then the projection* Q *given by* (1.1) *is unique.*

It is worth saying that in the cases of $p = 1$ and $p = \infty$ the projection Q given by (1.1) is not unique (for $m, n \geqslant 3$).

THEOREM 2.3. *If* $X = (M(n, m), \|\cdot\|_1)$ *or* $X = (M(n, m), \|\cdot\|_\infty)$ *then the projection* Q *given by* (1.1) *is not unique* (*for* $m, n \geqslant 3$).

This follows from the dimensional computation presented (in sketch) below for $X = (M(n, m), \|\cdot\|_1)$ (for $X = (M(n, m), \|\cdot\|_\infty)$ reasoning is almost the same).

Let C be a convex subset of a Banach space X. A point $e \in C$ is called an extreme point of C if e is not a center of any non-degenerate line segment in C (i.e., for any $x, y \in C$ if $e = \lambda x + (1 - \lambda) y$ then $\lambda = 0$ or $\lambda = 1$). The set of all extreme points of C is denoted by Ext_C. If X is a Banach space then $\mathrm{Ext}_X := \mathrm{Ext}_{B_X}$, where B_X is a unit ball in X.

Take $P := Q + \alpha L$, where $L: X \to W$ is such that $L/_W = 0$ and $\alpha > 0$. Since Q is a projection from X to W, P is also a projection from X to W. Our aim is to choose L and α in such way that $\|P\| = \|Q\|$. It is well known that there are $(y, x) \in \mathrm{Ext}_{X^*} \times \mathrm{Ext}_X$ such that $y(Px) = \|P\|$ (compare this with Definition 1.3). Let then

$$\tilde{\mathscr{E}}_1(P) = \{(x, y) \in \mathrm{Ext}_X \times \mathrm{Ext}_{X^*} : |y(Px)| = \|P\|\}.$$

Now, we will construct L. Take e_{rs} ($e_{rs}(i, j) = \delta_{ri}\delta_{sj}$). These elements are extreme points of X and form a basis of X. By the form of extremal points of a unit sphere in ℓ^∞ and the formula (1.1) of the projection Q for e_{rs} there could be only one $y_{rs} \in \mathrm{Ext}_{X^*}$ such that $y_{rs}(Q(e_{rs})) = \|Q\|$ (since the representation of $Q(e_{rs})$ in base e_{ij} ($i = 1, ..., n$, $j = 1, ..., m$) has all non-zero coefficients).

Any operator $L: X \to X$ can be written in form

$$L(e_{rs}) = \sum_{kl} a_{kl}^{rs} e_{kl}. \tag{2.9}$$

If we assume that $L: X \to W$ then (since $\dim W = n + m - 1$) there are only $(n + m - 1)\, nm$ independent a_{kl}^{rs} in the formula (2.9). Assuming that $y_{rs}(Le_{rs}) = 0$ gives us additionally nm linear equations on a_{kl}^{rs} and the assumption $L/_W = 0$ gives $(n + m - 1)(n + m - 1)$ linear equations on a_{kl}^{rs}.

Since $(n+m-1)\,nm - nm - (n+m-1)(n+m-1) > 0$ (for $m,\ n \geqslant 3$) we can choose the numbers a^{rs}_{kl} such that

$$L\colon X \to W, \quad L \neq 0 \quad \text{and} \quad L/_W = 0 \quad \text{and} \quad y_{rs}(Le_{rs}) = 0. \tag{2.10}$$

Since, in our case, the set $\mathrm{Ext}_{X^*} \times \mathrm{Ext}_X$ is finite

$$M = \max\{|y(Qx)|, \text{ where } (x,\ y) \in (\mathrm{Ext}_X \times \mathrm{Ext}_{X^*}) \backslash \tilde{\mathscr{E}}_1(Q)\} < \|Q\|. \tag{2.11}$$

Take α such that

$$\|\alpha L\| < \|Q\| - M. \tag{2.12}$$

Now, for $(x,\ y) \in \tilde{\mathscr{E}}_1(Q)$, by (2.10) we get

$$|y((Q+\alpha L)\,x)| = |y(Qx) + \alpha y(Lx)| = |y(Qx)| = \|Q\|,$$

and for $(x,\ y) \in (\mathrm{Ext}_X \times \mathrm{Ext}_{X^*}) \backslash \tilde{\mathscr{E}}_1(Q)$, by (2.11) and (2.12) we get

$$\begin{aligned} |y((Q+\alpha L)\,x)| &\leqslant |y(Qx)| + |y(\alpha Lx)| \leqslant M + \|\alpha L\| \\ &< M + \|Q\| - M = \|Q\|. \end{aligned}$$

Using the above estimations and the well known fact

$$\|P\| = \max\{|y(Px)|, \text{ where } (x,\ y) \in \mathrm{Ext}_X \times \mathrm{Ext}_{X^*}\},$$

yields $\|Q + \alpha L\| = \|Q\|$. Therefore $Q + \alpha L$ is also a minimal projection.

This remark shows that the assumption of smoothness of the considered space is essential and cannot be omitted. Below we present more general spaces (i.e., Orlicz spaces) fulfilling Condition 0.3.

Let φ be a convex function such that $\varphi(0) = 0$ and $\varphi/_{(0,\ \infty)} > 0$. A function with the above properties will be called an *Orlicz function.*

If the Orlicz function φ satisfies the conditions

$$\lim_{x \to 0^+} \frac{\varphi(x)}{x} = 0 \qquad \text{and} \qquad \lim_{x \to +\infty} \frac{\varphi(x)}{x} = +\infty,$$

then we will called it briefly *N-function.* From now on, we will only consider N-functions.

Let p be the right derivative of φ. The function

$$q(s) = \sup\{t : p(t) \leqslant s\} = \inf\{t : p(t) > s\}$$

we call the right-inverse function of p.

Let φ be an N-function, p be the right derivative of φ, and q be the right-inverse function of p. Then we call

$$\psi(v) = \int_0^v q(s)\,ds$$

the complementary function of φ.

The Orlicz modular corresponding to the function φ is defined by

$$\rho_\varphi(x) = \sum_{i,j} \varphi(|x_{ij}|),$$

for any $x \in M(n, m)$ and $x = \sum_{i,j} x_{i,j} e_{i,j}$.

The Luxemburg norm and the Orlicz norm is defined respectively by

$$\|x\|_{\rho_\varphi} = \inf\{d > 0 : \rho_\varphi(x/d) \leqslant 1\}, \tag{2.13}$$

$$|||x|||_{\rho_\varphi} = \sup\left\{\sum_{i,j} y_{ij} x_{ij} : \rho_\psi(y) \leqslant 1\right\}. \tag{2.14}$$

Using Amemiya formulas we get

$$\|x\|_{\rho_\varphi} = \inf_{d>0}\{d \max\{1, \rho_\varphi(x/d)\}\}, \tag{2.15}$$

$$|||x|||_{\rho_\varphi} = \inf_{d>0}\{d + d\rho_\varphi(x/d)\}. \tag{2.16}$$

For basic facts concerning Orlicz spaces and extensions the reader is referred to [10] and [20].

It is well known that there are sufficient and necessary conditions in terms of function φ for Orlicz spaces to be smooth, even in cases more general than presented below (see, e.g., [10, Chapter 2.7]).

The function φ is called smooth if its right derivative p is continuous.

Let

$$\pi_\varphi(\alpha) = \inf\{t > 0 : \psi(p(t)) \geqslant \alpha\},$$

then in our cases these conditions can be formulated as follows

THEOREM 2.4 [10]. *Let φ be an N-function. Then $M(n, m)$ with the Luxemburg norm is a smooth space if and only if φ is smooth on* $(0, \varphi^{-1}(1))$.

THEOREM 2.5 [10]. *Let φ be an N-function. Then $M(n, m)$ with the Orlicz norm is a smooth space if and only if φ is smooth on* $(0, \pi_\varphi(1/2))$ *and* $p_-(\pi_\varphi(1/2)) = \psi^{-1}(1/2)$. ($p_-(a)$ *means* $\lim_{t\to a^-} p(t)$.)

Simple computation shows that in $M(n, m)$ with the Luxemburg or the Orlicz norm the transformations A_π are isometries. Therefore we can state the following

THEOREM 2.6. *Let φ be an N-function such that φ is smooth on $(0, \varphi^{-1}(1))$. Consider $M(n, m)$ with the Luxemburg norm, then the projection Q given by* (1.1) *is unique.*

THEOREM 2.7. *Let φ be an N-function such that φ is smooth on $(0, \pi_\varphi(1/2))$ and $p_-(\pi_\varphi(1/2)) = \psi^{-1}(1/2)$. Consider $M(n, m)$ with the Orlicz norm, then the projection Q given by* (1.1) *is unique.*

It can be easily seen that the functions $\varphi(t) = t^p$, where $p \in (1, \infty)$ fulfills the conditions required on the function φ in Theorem 2.6 and Theorem 2.7.

EXAMPLE 2.8. If we take $\varphi(t) = e^t - t - 1$ then the Orlicz norm and the Luxemburg norm generated by so chosen φ are smooth. Moreover these norms are of course different from p-norms.

Now, we present another example of norms fulfilling Condition 0.3.

Let E be a convex symmetric body in $\mathbb{R}^{nm}$ (i.e., E is convex, compact with nonempty interior and $E = -E$). Let E^* denote its polar

$$E^* = \{x \in \mathbb{R}^{nm} : x \cdot y \leqslant 1, \text{ for any } y \in E\}.$$

Here "$\cdot$" denotes the canonical inner product in $\mathbb{R}^{nm}$. Define

$$f(x) := \sup\{|x \cdot y| : y \in E^*\} \qquad \text{for} \quad x \in \mathbb{R}^{nm} \tag{2.17}$$

and

$$f^*(x) := \sup\{|x \cdot y| : y \in E\} \qquad \text{for} \quad x \in \mathbb{R}^{nm}. \tag{2.18}$$

Then both functions f and f^* are norms in $\mathbb{R}^{nm}$ and E, E^* are the unit balls for f and f^*, respectively. For $k \in \mathbb{N}$ put

$$f_k(x) = \left(\frac{1}{vol(E^*)} \int_{E^*} (x \cdot y)^{2k}\, dy\right)^{1/2k}.$$

Then, by [1, Remark 1.7], f_k are increasing sequence of norms such that $f_k \to f$ (i.e., $f_k(x) \to f(x)$ for any $x \in \mathbb{R}^{nm}$) and $f_k \in \mathscr{C}^\infty(\mathbb{R}^{nm} \setminus \{0\})$. Therefore f_k are also smooth norms. Since, by [1, Proposition 1.8], if A_π is an isometry in the norm f then it is also an isometry in the norm f_k we may state the following

THEOREM 2.9. *Assume that for any $\pi \in S_n \times S_m$ the transformation A_π is an isometry in the given norm f. Consider $M(n, m)$ with f_k norm, then the projection Q given by* (1.1) *is unique.*

One can easily seen that if E fulfills the condition

$$\text{for any } \pi \qquad A_\pi(y) \in E \quad \text{provided} \quad y \in E; \tag{2.19}$$

then A_π are isometries in the norm f. Therefore this class greatly extends our previous examples.

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REFERENCES

1. M. Baran, Conjugate norms on $\mathbb{C}^n$ and related geometrical problems, *Diss. Math.* **CCCLXXVII** (1998), 1–67.
2. D. L. Berman, On the impossibility of constructing a linear polynomial operator furnishing an approximation of the order of best approximation, *Dokl. Akad. Nauk. SSSR* **120** (1958), 143–148.
3. B. L. Chalmers and G. Lewicki, Symmetric subspaces of ℓ_1 with large projection constants, *Studia Math.* **134** (1999), 119–133.
4. B. L. Chalmers and G. Lewicki, Minimal projections onto some subspaces of ℓ_1^n, *Funct. Approx.* **XXVI** (1998), 85–92.
5. B. L. Chalmers and G. Lewicki, Two-dimensional real symmetric spaces with maximal projection constant, *Ann. Polon. Math.* **LXXIII.2** (2000), 119–134.
6. B. L. Chalmers and F. T. Metcalf, Determination of a minimal projection from $C[-1, 1]$ onto the quadratics, *Numer. Funct. Anal. Optim.* **11** (1990), 1–10.
7. B. L. Chalmers and F. T. Metcalf, The determination of minimal projections and extensions in L^1, *Trans Amer. Math. Soc.* **329** (1992), 289–305.
8. B. L. Chalmers and F. T. Metcalf, A characterization and equations for minimal projections and extensions, *J. Oper. Theory* **32** (1994), 31–46.
9. B. L. Chalmers and F. T. Metcalf, The minimal projection from L^1 onto π_n, *in* "Approximation Theory, VII" (C. K. Chui and L. Schumaker, Eds.), pp. 61–69, Dekker, New York, 1996.
10. S. Chen, Geometry of Orlicz spaces, *Diss. Math.* **CCCLVI** (1996), 1–204.
11. E. W. Cheney, C. R. Hobby, P. D. Morris, F. Schurer, and D. E. Wulbert, On the minimal property of the Fourier projection, *Trans. Amer. Math. Soc.* **143** (1969), 249–258.
12. E. W. Cheney and W. A. Light, "Approximation Theory in Tensor Product Spaces," Lecture Notes in Math., Vol. 1169, Springer-Verlag, Berlin, 1985.
13. S. D. Fisher, P. D. Morris, and D. E. Wulbert, Unique minimality of Fourier projections, *Trans. Amer. Math. Soc.* **265** (1981), 235–246.

14. P. Habala, P. Hajek, and V. Zizler, "Introduction to Banach Spaces I, II," Praha, 1996.
15. E. J. Halton and W. A. Light, Minimal projections in L_p-spaces, *Math. Proc. Cambridge Philos. Soc.* **97** (1985), 127–136.
16. P. V. Lambert, Minimum norm property of the Fourier projection in spaces of continuous functions, *Bull. Soc. Math. Belg.* **21** (1969), 359–369.
17. P. V. Lambert, Minimum norm property of the Fourier projection in L^1-spaces, *Bull. Soc. Math. Belg.* **21** (1969), 370–391.
18. G. Lewicki, G. Marino, and P. Pietramala, Fourier-type minimal extensions in real L^1-space, *Rocky Mountain J. Math.*, to appear.
19. W. A. Light, Minimal projections in tensor-product spaces, *Math. Z.* **191** (1986), 633–643.
20. J. Musielak, "Orlicz Spaces and Modular Spaces," Lecture Notes in Math., Vol. 1034, Springer-Verlag, New York/Berlin, 1988.
21. W. Odyniec and G. Lewicki, "Minimal Projections in Banach Spaces," Lecture Notes in Math., Vol. 1449, Springer-Verlag, Berlin, 1990.
22. M. Prophet, Codimension one minimal projections onto the quadratics, *J. Approx. Theory* **81** (1995), 27–42.
23. P. Wojtaszczyk, "Banach Spaces For Analysis," Cambridge Univ. Press, Cambridge, UK, 1991.